Measuring Stochastic Long-Range Dependence

Calculating the Hurst Exponent of the S&P 500

Ajay Dugar

Departments of Economics and Mathematics

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Outline

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Common Assumptions

The Foundation of Modern Financial Theory

What are the foundational assumptions when it comes to market research, and where do they come from?

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- With these findings, assuming common rationality, all asset prices reflect complete information, i.e. EMH (Fama)

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1. Economic Assumptions
   - Common Assumptions
   - Initial Analysis
2. Findings
   - Distributions
   - Calculations
3. Summary

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- Where do we go from here? **Fractal analysis**

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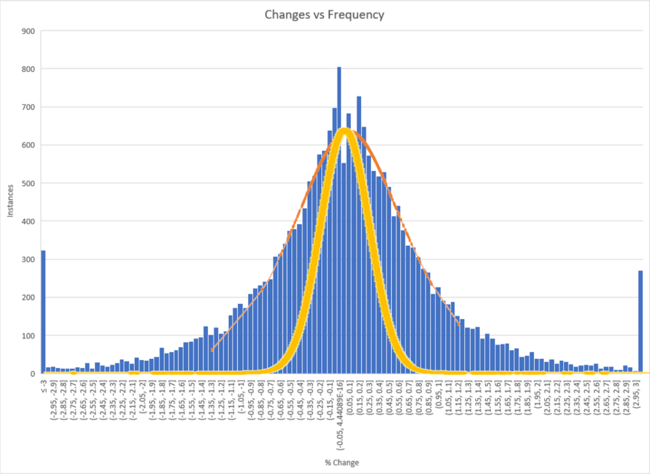
- Taking a look at real data, we'll test this assumption and find a better fit

Choosing a Distribution



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- **Thus, the Cauchy distribution is a better fit for the long, fat tailed data**

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Fractal Dimension

$$C_H^d(S) := \inf\left\{\sum_i r_i^d : \text{ there is a cover of } S \text{ by balls with radii } r_i > 0\right\}$$

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- Essentially, this dimension tells us how spaces scale

Scaling


S&P 500 Closing Price
INDEX PRICE (USD)
3000
2500
2000
1500
1000
500
0
'27 '30 '32 '35 '38 '41 '43 '46 '49 '52 '54 '57 '60 '62 '65 '68 '71 '73 '76 '79 '82 '84 '87 '90 '93 '95 '98 '01 '04 '06 '09 '12 '14 '17
YEAR

Scaling


S&P 500 Closing Price
4096
2048
1024
512
256
128
64
32
16
8
4
2
1
INDEX PRICE (USD)
'27 '30 '32 '35 '38 '41 '43 '46 '49 '52 '54 '57 '60 '62 '65 '68 '71 '73 '76 '79 '82 '84 '87 '90 '93 '95 '98 '01 '04 '06 '09 '12 '14 '17
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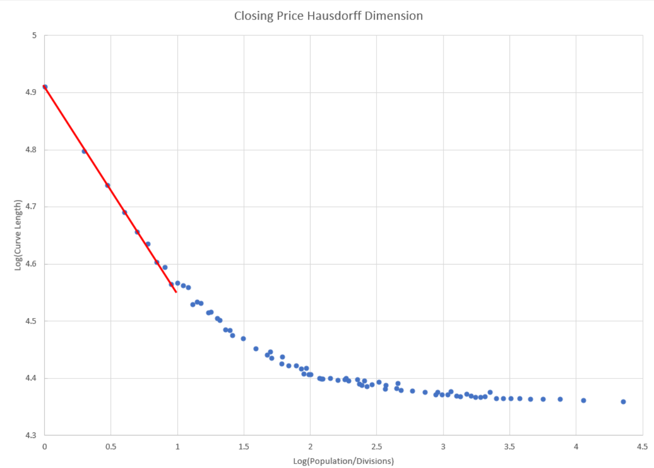

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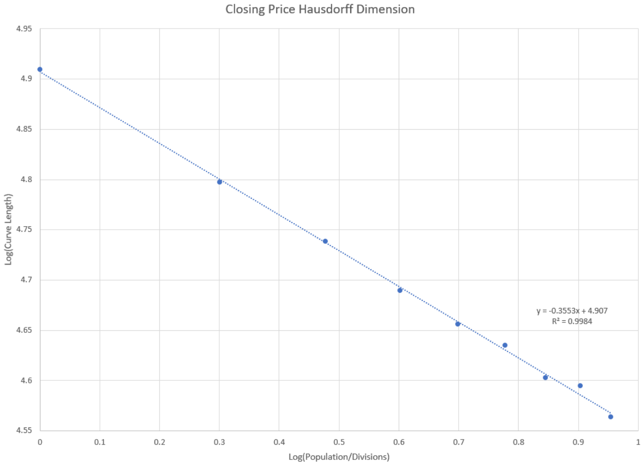

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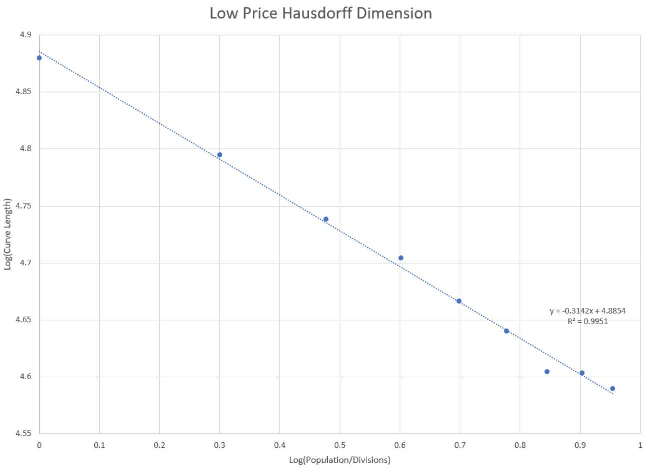

Hausdorff Dimension and Hurst Exponent

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- This value corresponds to a Hurst exponent of 0.6447 (H_{high} = 0.7151, H_{low} = 0.6858)
- Comparing this to the Hurst exponent value of the closing price computed by Bayraktar, et. al, of 0.6156 ± 0.0531, we see that these results are consistent

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- **Markets are more erratic and random than conventional wisdom suggests**
- Next Steps
  - Increase the data set size to improve accuracy of these findings
  - Apply this curve fitting algorithm to different market data to determine better measures of volatility and risk

References

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